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O‘ZGARUVCHAN KOEFFITSIYENTLI KLEIN–GORDON TENGLAMASI UCHUN YUQORI ANIQLIKDAGI AYIRMALI SXEMANING TURG‘UNLIGI VA YAQINLASHISHI

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Annotatsiya. Xususiy hosilali differensial tenglamalar, xususan, Klein–Gordon tenglamasi relyativistik kvant mexanikasi va to‘lqinlar dinamikasida keng qo‘llaniladi. Ushbu ishning maqsadi chekli sohada o‘zgaruvchan koeffitsiyentli Klein–Gordon tenglamasi uchun $O(\tau^2 + h^4)$ aniqlikdagi ayirmali sxema qurish hamda uning turg‘unligi va yaqinlashishini nazariy isbotlashdan iborat.

Kalit so‘zlar: ayirmali sxema, Klein–Gordon tenglamasi, Koshi–Shvars tengsizligi, Gronuoll lemmasi, turg‘unlik va yaqinlashish.

O‘zgaruvchan koeffitsiyentli Klein–Gordon tenglamasi

$\overline{Q}_T = \{(x,t): 0 \quad x \quad l, 0 \quad t \quad T\}$ sohada o‘zgaruvchan koeffitsiyentli Klein–Gordon tenglamasi uchun boshlang‘ich-chegaraviy masalani qaraymiz:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \quad k(x,t)\frac{\partial u}{\partial x} \quad - mu + f(x,t), \quad m = const > 0, \tag{1}$$

$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = \overline{v}_0(x), \tag{2}$$

$$u(0,t) = \mu_1(t), \quad u(l,t) = \mu_2(t), \tag{3}$$

bu yerda $0 < k_1 \quad k(x,t) \quad k_2, u(x,t) \quad C^{4,6}\left(\overline{Q}_T\right), p \quad C^{0.5}\left(\overline{Q}_T\right)$ va $f \quad C^{0.4}\left(\overline{Q}_T\right)$.

Boshlang‘ich differensial masalani $\overline{\omega}$ to‘rda ayirmali sxema bilan approksimatsiyalaymiz:

$$y_{\overline{t}t} = \Lambda y^{(\sigma,\sigma)} - \frac{h^2}{12}\Lambda(py_{\overline{t}t}) - m \quad y^{(\sigma,\sigma)} + \frac{h^2}{12}\Lambda(py^{(\sigma,\sigma)}) \quad + \varphi, (x,t) \quad \omega_h \quad \omega_\tau,$$

(4)

$$y(x,0) = u_0(x), \quad x \quad \overline{\omega}_h, \quad y_t(x,0) = u_1(x), \quad x \quad \omega_h, \tag{5}$$

$$y(0,t) = \mu_1(t), \quad y(l,t) = \mu_2(t), \quad t \quad \omega_\tau, \tag{6}$$

bu yerda

$$\Lambda y = \left(a(x,t_n)y_{\overline{x}}\right)_x, \quad \sigma = 0.5 \quad \varphi = f + \frac{h^2}{12}\Lambda(pf), \quad p(x,t) = \frac{1}{k(x,t)},$$

$$a(x,t) = 6 \quad p(x-h,t) + 4p \quad x - \frac{h}{2}, t \quad + p(x,t) \quad^{-1}, \quad 0 < c_1 \quad a(x,t) \quad c_2,$$

$$u_1(x) = \overline{v}_0(x) + \frac{\tau}{2}\left[(k(x,0)u\,(x,0)) - mu(x,0) + f(x,0)\right], \quad x \quad \omega_h.$$

(4) – (6) ayirmali sxema (1) – (3) differensial masalani fazo bo‘yicha to‘rtinchi, vaqt bo‘yicha esa ikkinchi tartibli qilib approksimatsiyalanishini osongina ko‘rsatish mumkin, ya’ni

$$\psi = -u_{\bar{t}\bar{t}} + \Lambda u^{(\sigma,\sigma)} - \frac{h^2}{12}\Lambda(pu_{\bar{t}\bar{t}}) - m\ u^{(\sigma,\sigma)} + \frac{h^2}{12}\Lambda(pu^{(\sigma,\sigma)}) + \varphi$$

va ikkinchi boshlang‘ich shart quyidagi aprior baholarni qanoatlantiradi:

$$\|\psi\| \quad M(h^4+\tau^2), \qquad M = const > 0,$$

$$\|\overset{\circ}{\psi}\| = \|u_1 - u_t^0\| \quad M_1\tau^2, \ M_1 = const > 0.$$

Ayirmali sxemaning tahlilini soddalashtirish maqsadida, biz faqat $k = k(x)$ koeffitsiyenti fazoviy koordinataga bog‘liq bo‘lgan holatni ko‘rib chiqamiz. (4) – (6) ayirmali sxema asosida olingan, ya’ni o‘ng tomoni $\tilde{f}$ hamda boshlang‘ich shartlari $\tilde{u}_0$ va $\tilde{u}_1$ ko‘rinishida g‘alayonlangan masalaning yechimini $\tilde{y}$ ko‘rib chiqamiz. Bunda aniq yechim va g‘alayonlangan yechim orasidagi farq, ya’ni $\overline{y} = \tilde{y} - y$ uchun quyidagi masala hosil bo‘ladi:

$$\overline{y}_{tt} = \Lambda\overline{y}^{(\sigma,\sigma)} - \frac{h^2}{12}\Lambda\left(p\overline{y}_{tt}\right) - m\ \overline{y}^{(\sigma,\sigma)} + \frac{h^2}{12}\Lambda\left(p\overline{y}^{(\sigma,\sigma)}\right) + \overline{\varphi}, \ (x\ t) \quad \omega_h \quad \omega_\tau \quad ,$$

(7)

$$\overline{y}(x,0) = \overline{u}_0(x), \ x \quad \overline{\omega}_h, \ \overline{y}_t(x,0) = \overline{u}_1(x), \ x \quad \omega, \tag{8}$$

$$\overline{y}(0,t) = 0, \ \overline{y}(l,t) = 0, \ t \quad \omega_\tau, \tag{9}$$

bu yerda $\overline{u}_0 = \tilde{u}_0 - u_0$, $\overline{u}_1 = \tilde{u}_1 - u_1$ va $\overline{\varphi} = \tilde{\varphi} - \varphi$.

Mazkur ayirmali masala chiziqli bo‘lsa-da, u Samarskiyning uch bosqichli operator-ayirmali sxemalar nazariyasi shartlarini qanoatlantirmaydi, shu sababli ushbu nazariyani unga nisbatan qo‘llab bo‘lmaydi. Biz energiya tengsizliklari usulidan foydalanamiz; ayirmali sxemalar nazariyasidan ba’zi ma’lum faktlarga, masalan, birinchi ayirmali Grin formulasi, ε-li Koshi–Shvars tengsizligi hamda Gronuoll lemmasining ayirmali analogiga va qo‘shimcha ravishda bizga quyidagi tasdiq ham kerak bo‘ladi.

1-lemma. Agar

$$h \quad h_0, \quad h_0 = \sqrt{\frac{3k_1}{m}}, \quad \tau \quad \frac{2h}{\sqrt{3k_1}} \tag{10}$$

shartlar bajarilsa, quyidagi

$$Q^n = \|\overline{y}_{\bar{t}}\|^2 + \frac{1}{2}\left(a,\overline{y}_x^2 + \overline{\overline{y}}_x^2\right. + \frac{m}{2}\left(\|\overline{y}\|^2 + \|\overline{\overline{y}}\|^2\right) - \frac{h^2}{12}\left(ap_{(-1)},\overline{y}_{\bar{x}\bar{t}}^2\right. - \frac{mh^2}{24}\left(ap_{(-1)},\overline{y}_{\bar{x}}^2 + \overline{\overline{y}}_{\bar{x}}^2\right.$$

ifoda o‘rinli bo‘ladi va $Q^n \quad 0$, bunda $p_{(-1)} = p_{i-1}$.

$\overline{y}$ uchun aprior baho olish maqsadida (7) ayirmali tenglamani $2\tau y_{\circ}$ ga skalyar ko‘paytiramiz va birinchi Grin ayirmali formulasini qo‘llab, quyidagi energiya munosabatini hosil qilamiz:

$$Q^{n+1} - \frac{h^2}{12}\left(ap_{\bar{x}}\left(\overline{y}_t - \overline{y}_{\bar{t}}\right), \overline{y}_{\bar{x}t} + \overline{y}_{\bar{x}\bar{t}}\right. \quad = \tag{11}$$

$$= Q^n + \frac{m\tau h^2}{12}\left(ap_{\bar{x}}\left(\overline{\overline{y}} + \overline{\overline{y}}\right), \overline{y}_{\circ\, t\bar{x}}\right) + 2\tau\left(\overline{\varphi}, \overline{y}_{\circ\, t}\right).$$

(11) ifodadagi Q_n va Q_{n+1} dan boshqa hadlarni ko‘rib chiqamiz. ε ga nisbatan Koshi-Shvarts tengsizligini qo‘llab quyidagi baholarni osongina hosil qilamiz:

$$-\frac{h^2}{12}\left(ap_{\bar{x}}\left(\overline{y}_t - \overline{y}_{\bar{t}}\right), \overline{y}_{\bar{x}t} + \overline{y}_{\bar{x}\bar{t}}\right) \quad -ch\left(\left\|\overline{y}_t\right\|^2 + \left\|\overline{y}_{\bar{t}}\right\|^2\right), \tag{12}$$

$$\frac{m\tau h^2}{12}\left(ap_{\bar{x}}\left(\overline{\overline{y}} + \overline{\overline{y}}\right), \overline{y}_{\circ\, t\bar{x}}\right) \quad ch\ \frac{m}{2}\left(\left\|\overline{\overline{y}}\right\|^2 + \left\|\overline{y}\right\|^2\right) + \frac{m}{2}\left(\left\|\overline{y}\right\|^2 + \left\|\overline{\overline{y}}\right\|^2\right), \tag{13}$$

$$2\tau\left(\overline{\varphi}, \overline{y}_{\circ\, t}\right) \quad \varepsilon\tau\left\|\varphi\right\|^2 + \frac{\tau}{2\varepsilon}\left(\left\|\overline{y}_t\right\|^2 + \left\|\overline{y}_{\bar{t}}\right\|^2\right), \tag{14}$$

bu yerda $c > 0$ soni m, ε va $\max\limits_{x\ \bar{\omega}_h}\left|p_{\bar{x}}(x)\right|$ qiymatlariga bog‘liq bo‘lgan konkret holat uchun xos o‘zgarmas miqdordir.

(11) va (13)-(14) tengsizliklarni inobatga olib, (10) shartlar bajarilganda quyidagi rekurrent munosabatga kelamiz:

$$Q^{n+1} \quad \left(1 + \tau c\right)Q^n + \tau c\left\|\overline{\varphi}\right\|^2 \quad e^{c\,\tau}Q^n + \tau c\left\|\overline{\varphi}\right\|^2 \tag{15}$$

Shunday qilib, quyidagi teorema o‘rinli.

1-teorema. Faraz qilaylik

$$\tau \quad \max\ 1, \sqrt{\frac{2}{3k_1}}\ \ h$$

sharti bajarilsin. U holda

$$Q^{n+1} \quad e^{ct_n}\ \ Q^1 + c\sum_{k=1}^{n}\tau\left\|\overline{\varphi}^k\right\|^2\ \ , \tag{16}$$

baholash hosil bo‘ladi, bu esa (4) – (6) ayirmali sxema yechimining boshlang‘ich ma’lumotlar va o‘ng tomonga nisbatan $L_2\left(\omega_h\right), W_2^1\left(\omega_h\right)$ va $C\left(\overline{\omega}_h\right)$ to‘r normalarida ρ turg‘unligini anglatadi.

Teoremaning isboti (6) tengsizlik va Gronuoll lemmasida kelib chiqadi, ya’ni

$$\left\|\overline{y} - y\right\| c \quad \frac{\sqrt{l}}{2}\left\|\tilde{y}_{\bar{x}} - y_{\bar{x}}\right\|.$$

(4) – (6) ayirmali tenglamalardagi y o‘rniga $z + u$ ni qo‘yib, z uchun quyidagi ifodani hosil qilamiz:

$$z_{\tilde{t}t} = \Lambda z^{(\sigma,\sigma)} - \frac{h^2}{12}\Lambda\left(pz_{\tilde{t}t}\right) - m\ \ z^{(\sigma,\sigma)} + \frac{h^2}{12}\Lambda\left(pz^{(\sigma,\sigma)}\right) + \psi,\ \ (x,t) \quad \omega_h \quad \omega_\tau$$

(17)

$$z\left(x,0\right) = 0,\ \ x \quad \overline{\omega}_h,\ \ z_t(x,0) = \overset{\circ}{\psi},\ \ x \quad \omega_h,\ \ \overset{\circ}{\psi} = O\left(\tau^2\right), \tag{18}$$

$$z(0,t) = 0,\ \ z(l,t) = 0,\ \ t \quad \omega_\tau, \tag{19}$$

bu yerda u funksiyasi (1) – (3) tenglamaning yechimi.

(17) – (19) va (7) – (9) masalalar bir xil. Shu sababli, xatolikni baholash uchun 1-teoremani qo‘llashimiz mumkin. U holda, (7) tengsizlikka muvofiq quyidagi bahoni olamiz:

$$\|z\|_C^2 \quad M_1 \left[\|\overset{\circ}{\psi}\|^2 + \frac{4c_2}{c^2}\|\overset{\circ}{\psi}_{\bar{x}}\|^2 + \frac{4m}{c^2}\|\overset{\circ}{\psi}\|^2 + cT \max_{t\ \omega_\tau} \|\psi(t)\|^2 \right] ,$$

bu yerda $M_1 = const > 0$.

Shunday qilib, endi yaqinlashish teoremasini ta’riflash imkoniyatiga ega bo‘ldik.

2-**teorema.** 1-teoremaning shartlari bajarilgan bo‘lsin. U holda (4) – (6) ayirmali sxemaning yechimi $C(\bar{\omega}_h)$ to‘r normasida (1) – (3) differensial masalaning aniq yechimiga yaqinlashadi va yechim uchun quyidagi xatolik bahosi o‘rinli bo‘ladi:

$$\left\|y^n - u^n\right\|_C \quad M_2\left(h^4 + \tau^2\right), \ \ n = 0,...,N_0,$$

bu yerda $M_2 = const > 0$.

Foydalanilgan adabiyotlar ro‘yxati

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