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SOURCE IDENTIFICATION PROBLEMS FOR SUBDIFFUSION EQUATIONS: SPACE-TIME DEPENDENCE, INTEGRAL OVERDETERMINATION AND NONLINEAR CASES

Abstract: We will consider the inverse problem of determining the right-hand side in diffusion equations using different overdetermination conditions. The novelty of the work is that the unknown functions depend on both time and part of the spatial variables. Only a few works are known where similar issues are studied using different methods. We will dwell on the difference between our results and the known ones later.

Key words: Subdiffusion equations; inverse problem; uniqueness and existence of solution; weak; strong; local and global solutions; Fourier method.

MSC2020: 35R11; 34A12; 74S25.

Аннотация: В работе рассматривается обратная задача определения правой части в диффузионных уравнениях при использовании различных условий переопределения. Новизна исследования заключается в том, что неизвестные функции зависят как от времени, так и от части пространственных переменных. В настоящее время известно лишь небольшое число работ, в которых аналогичные задачи исследуются с

https://wordlyknowledge.uz/index.php/gpble
Qabul qilingan sana: 07.12.2025
Nashr etilgan sana: 30.01.2026

применением различных методов. Отличия полученных в данной работе результатов от известных будут подробно обсуждены далее.

Ключевые слова: уравнения субдиффузии; обратная задача; единственность и существование решения; слабые и сильные решения; локальные и глобальные решения; метод Фурье.

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Annotatsiya: Ushbu ishda turli shartlar asosida diffuziya tenglamalarida o‘ng tomonini aniqlashga doir teskari masala ko‘rib chiqiladi. Tadqiqotning yangiligi shundaki, noma’lum funksiyalar vaqtga hamda fazoviy o‘zgaruvchilarning bir qismiga bog‘liq holda qaraladi. Hozirgi kunga qadar bunday masalalar turli usullar yordamida o‘rganilgan ishlar juda kam sonni tashkil etadi. Mazkur ishda olingan natijalarning ma’lum natijalardan farqi keng batafsil yoritiladi.

Kalit so‘zlar: subdiffuziya tenglamalari; teskari masala; yechimning yagonaligi va mavjudligi; kuchsiz va kuchli yechimlar; lokal va global yechimlar; Furye usuli.

MSC2020: 35R11; 34A12; 74S25.

1 Introduction

The present paper is based on the following works of ours [1]-[3]. Let us dwell in detail on work [1] and then list the results of the other two works.

Let $Q=(0,T)\times G_x\times\Omega_y$ and $D=G_x\times\Omega_y$, where $G:=G_x\subset\mathbb{R}^m$ and $\Omega:=\Omega_y\subset\mathbb{R}^n$ are bounded domains, with a sufficiently smooth boundary. Consider the following initial-boundary value problem:

$$\begin{cases} D_t^\alpha u-\Delta_x u-\Delta_y u=g(t,x,y)+f(t,x,y)\cdot h(t,x), \quad (t,x,y)\in Q, \\ u(0,x,y)=\varphi(x,y), \quad (x,y)\in D, \\ u(t,x,y)=0, \quad t\in[0,T], \quad (x,y)\in\partial D. \end{cases} \tag{1}$$

https://wordlyknowledge.uz/index.php/gpble
Qabul qilingan sana: 07.12.2025
Nashr etilgan sana: 30.01.2026

Here Δ_x and Δ_y are the Laplace operators in variables x and y respectively, f,g and φ are given functions, $\alpha\in(0,1)$, and D_t^α is the fractional Gerasimov-Caputo derivative. Recall, if $v(t)$ is an absolutely continuous function, then its Gerasimov-Caputo derivative has the form

$$D_t^\alpha v(t)\equiv J_t^{1-\alpha}D_t v(t),\quad J_t^\alpha v(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}v(\tau)\,d\tau,$$

where $D_t=d/dt$, $\Gamma(\alpha)$ is the gamma function, J_t^α is the Riemann-Liouville fractional integral.

If the function $h(t,x)$ is given, then the above problem is called **the forward problem** and its solution exists and is unique under certain conditions on the problem's data.

Now we will assume that the coefficient $h(t,x)$ of the source function is unknown and must be determined. The main goal of this talk is to study *the inverse problem* of determining a pair of functions $\{u(t,x,y),h(t,x)\}$ under the overdetermination condition specified by the following integral measurement:

$$\int_\Omega u(t,x,y)\,\omega(y)\,dy=\psi(t,x),\quad t\in[0,T],\quad x\in G, \tag{2}$$

where ω and ψ are known functions, the conditions on which we will determine later. If $h(t,x)$ depends only on t or only on (x,y), then quite a lot of works are known. In the case where the source function depends on both time and spatial variables, only a few works are known. Let us consider some of them in more detail. First, in all these works, the authors considered the following overdetermination condition:

$$u(t,x,l_0)=\psi(t,x),\quad t\in(0,T),\quad x\in G,\quad l_0\in(0,\pi), \tag{3}$$

where ψ is a known continuous function.

https://wordlyknowledge.uz/index.php/gpble
Qabul qilingan sana: 07.12.2025
Nashr etilgan sana: 30.01.2026

In [4], an inverse problem for a hyperbolic equation is considered when the right-hand side depends on both time and part of the spatial variable. The authors successfully established the uniqueness of the classical solution. In [5], the authors proposed a numerical method for solving a similar problem for a parabolic equation. For diffusion equations (i.e. $\alpha=1$) similar problems are solved (see [6] and [7]) for the case when the spatial domain $G_x \times \Omega_y$ is a half-space $\mathbb{R}\times\mathbb{R}_+$, and this fact plays an important role in proving the solvability of the inverse problem.

Let us also note the following two works [8], [9], where the inverse problem of simultaneously determining the right-hand side of the equations and some coefficients, which depend on time and on some of the spatial variables, is considered. The authors of [8] considered an equation with a fractional derivative $\alpha\in(0,2)$ in the domain $G_x\times(0,l)$, and the fact that the domain Ω_y is one-dimensional plays an important role in the reasoning. In addition, there are some conditions on the domain G_x. The inverse problem of determining a source that depends on both time and y was studied. Using the Fourier method (expanding in eigenfunctions of the entire elliptic part), the problem was reduced to an integral equation and then, using the method of compressed mappings, the existence and uniqueness of a solution to the inverse problem was proven. The authors also established some results on Lipschitz stability for restoring the source, which were then used to prove the stability of the reaction coefficient. In the paper [9] the authors study the simultaneous determination of three unknown coefficients: the coefficient of the time derivative and two lower coefficients. The unknown functions depend on (t,x), and the equation is considered in a two-dimensional space $x\in\mathbb{R}$, $y\in\mathbb{R}$. The authors, using overdetermination conditions, reduced the original inverse problem to a forward problem for a loaded equation (containing traces of unknown functions and their derivatives). The correctness of the resulting forward problem is studied using the weak approximation method.

https://wordlyknowledge.uz/index.php/gpble
Qabul qilingan sana: 07.12.2025
Nashr etilgan sana: 30.01.2026

In all the above works $y\in\mathbb{R}^1$: the study of the case $y\in\mathbb{R}^n$ is a little more complicated and, as will be shown in this report, with the growth of n, the requirement for the smoothness of the problem data increases.

Weak formulation. We multiply the equation in ((1)) by the function ω and integrate over the domain Ω. Then

$$D_t^\alpha\psi-\Delta_x\psi-(\Delta_y u,\omega)=(f(t,x,\cdot),\omega)\,h(t,x)+(g(t,x,\cdot),\omega).$$

Now we need the following conditions for the function ω:

$$\omega\in\dot{W}_2^1(\Omega)\ \text{ and }\ \int_\Omega \varphi(x,y)\,\omega(y)\,dy=\psi(0,x). \tag{4}$$

Taking into account this property, we get $(\Delta_y u,\omega)=-(\nabla_y u,\nabla\omega)$. Let $(f(t,x,\cdot),\omega)\neq 0$ for all $(t,x)\in[0,T]\times G$. Then

$$h(t,x)=\frac{D_t^\alpha\psi(t,x)-\Delta_x\psi(t,x)-(g(t,x,\cdot),\omega)+(\nabla_y u,\nabla\omega)}{(f(t,x,\cdot),\omega)}. \tag{5}$$

Definition 1.1 Find a pair of functions $\{u(t,x,y),h(t,x)\}$, where $h(t,x)$ has the form (5) and the function $u(t,x,y)$ satisfies the following conditions:

1. $u\in C([0,T];L_2(D))$, $\quad u\in L_2(0,T;\dot{W}_2^1(D))$;
2. $D_t^\alpha u\in L_2(0,T;L_2(D))$;
3. $u(0,x,y)=\varphi(x,y)$ a.e. in D;
4. For any $v\in\dot{W}_2^1(D)$ and almost every $t\in(0,T]$, the following equality holds:

$$\int_D D_t^\alpha uv\,dx\,dy+\int_D(\nabla_x u\nabla_x v+\nabla_y u\nabla_y v)\,dx\,dy=\int_D fhv\,dx\,dy+\int_D gv\,dx\,dy.$$

2 Eigenvalues of the Laplace Operator with Dirichlet Boundary Conditions.

https://wordlyknowledge.uz/index.php/gpble
Qabul qilingan sana: 07.12.2025
Nashr etilgan sana: 30.01.2026

In order to formulate the main result, it is necessary to recall some definitions and known results. The eigenvalue problem for the Laplace operator is given by:

$$-\Delta v=\lambda v \quad \text{in}\Omega:=\Omega_y\subset\mathbb{R}^n, \quad v=0 \quad \text{on}\partial\Omega. \tag{7}$$

This operator's spectrum consists of a sequence of positive eigenvalues $0<\lambda_1\leq\lambda_2\leq\lambda_3\leq\ldots$, counted with multiplicity, such that $\lambda_k\to+\infty$ as $k\to\infty$. Each eigenvalue λ_k corresponds to an eigenfunction $v_k(y)$, and the eigenfunctions $\{v_k(y)\}$ form an orthonormal basis in $L^2(\Omega)$.

The series $\sum_{k=1}^{\infty}\lambda_k^{-a}$ converges if $a>n/2$ (see, e.g., [10], p. 456). For any positive $\varepsilon>0$, we denote

$$C_\varepsilon:=\sum_{j=1}^{\infty}\lambda_j^{-\frac{n}{2}-\varepsilon}. \tag{8}$$

By Green's identity and since $\|v_k\|_{L^2(\Omega)}=1$, we have

$$\int_\Omega |\nabla v_k|^2\,dy=\lambda_k. \tag{9}$$

and

$$\int_\Omega \nabla v_k(y)\nabla v_j(y)\,dy=-\int_\Omega \Delta v_k(y)\,v_j(y)\,dy=\lambda_k\int_\Omega v_k(y)\,v_j(y)\,dy=\begin{cases}\lambda_k, & \text{if } j=k;\\ 0, & \text{if } j\neq k.\end{cases} \tag{10}$$

Let us define the degree of the Laplace operator using the spectral theorem:

$$(-\Delta)^\tau w(y)=\sum_{k=1}^{\infty}\lambda_k^\tau\, w_k\, v_k(y),\ \tau\geq 0,\ \|\Delta^\tau w\|^2_{L_2(\Omega)}:=\|w\|^2_\tau=\sum_{k=1}^{\infty}\lambda_k^{2\tau}\,|w_k|^2,$$

where w_k is a Fourier coefficient of $w(y)$. For the function w depending also on $x\in G$ and on $t\in[0,T]$, i.e. $w=w(t,x,y)$, we introduce the notation

$$\|w(t,\cdot,\cdot)\|^2_{\tau,G}:=\int_G\int_\Omega |(-\Delta)^\tau w(t,x,y)|^2\,dy\,dx=\int_G\sum_{k=1}^{\infty}\lambda_k^{2\tau}\,|w_k(t,x)|^2\,dx.$$

The two-parameter Mittag-Leffler function has the form:

https://wordlyknowledge.uz/index.php/gpble
Qabul qilingan sana: 07.12.2025
Nashr etilgan sana: 30.01.2026

$$E_{\rho,\mu}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\rho k+\mu)},\quad \rho>0,\quad \mu,z\in\mathbb{C}.$$

There is a constant $M:=M(b)$ such that $E_{\rho,\mu}(z)\leq M$, $\mu>0$, $z\in[0,b]$. Next, it is convenient for us to introduce a notation for some positive number ε:

$$2\tau:=2\tau(\varepsilon)=\frac{n}{2}+1+\varepsilon. \tag{11}$$

and (remember y is a n dimensional vector)

$$M_\alpha=\frac{MT^\alpha}{\alpha}. \tag{12}$$

Recall the constant C_ε is defined in ((8)).

3 Main Results

We now state the main result of this talk.

Theorem 1.1 *Let ε be any positive number and suppose the following conditions hold:*

1. $f(t,x,y)\in C(\overline{Q})$, $(f(t,x,\cdot),\omega)\neq 0$, and denote

$$f_0=\max_{t,x}\left|\frac{1}{(f(t,x,\cdot),\omega)}\right|;$$

2. $g(t,x,y)\in C(\overline{Q})$, and denote $g_0=\max_{t,x}|(g(t,x,\cdot),\omega)|$;

3. $D_t^\alpha\psi,\ \Delta_x\psi\in C([0,T]\times\overline{G})$, and denote

$$\psi_0=\max_{t,x}(|D_t^\alpha\psi|+|\Delta_x\psi|);$$

4. $2M_\alpha C_\varepsilon f_0^2\,||\nabla_y\omega||^2_{L_2(\Omega)}\max_{t,x}||f(t,x,\cdot)||^2_\tau\leq 1$, and denote

$$f^*=\max_t||f(t,\cdot,\cdot)||^2_{\tau,G};$$

5. $g^*=\max_t||g(t,\cdot,\cdot)||^2_{\tau,G}<\infty$;

https://wordlyknowledge.uz/index.php/gpble
Qabul qilingan sana: 07.12.2025
Nashr etilgan sana: 30.01.2026

6. $\|\nabla_x \varphi\|^2_{L_2(D)} < \infty$, and $\varphi^* = ||\varphi||^2_{\tau, G} < \infty$.

Then, the inverse problem has a unique weak solution. Moreover, the following estimates hold:

$$\max_{[0,T]} \|u(t,\cdot,\cdot)\|^2_{\tau,G} \le K_1, \tag{13}$$

$$\int_0^t \left(||\nabla_x u(s,\cdot,\cdot)||^2_{L_2(Q)} + ||\nabla_y u(s,\cdot,\cdot)||^2_{L_2(Q)} \right) ds \le K_2, \tag{14}$$

$$\int_0^t ||D_s^\alpha u(s,\cdot,\cdot)||^2_{L_2(Q)} ds \le K_3, \tag{15}$$

$$\max_{[0,T]} ||h(t,\cdot)||^2_{L_2(G)} \le K_4, \tag{16}$$

where K_j, $j=1,\cdots,4$, are positive constants depending on the data of the problem.

In our work [2], a similar problem (the unknowns are $u(t,x,y)$ and $h(t,x)$) with the overdetermination condition.

$$u(t,x,l_0) = \psi(t,x), \quad t \in (0,T), \quad x \in G, \quad l_0 \in (0,\pi), \tag{17}$$

instead of

$$\int_\Omega u(t,x,y)\,\omega(y)\,dy = \psi(t,x), \quad t \in [0,T], \quad x \in G, \ \omega \in \dot{W}_2^1(0,\pi), \tag{18}$$

is investigated. As noted above, in all previously known works, this inverse problem was considered with exactly this condition ((17)). As shown in the work [2], for the correctness of the problem with the condition ((17)), it is necessary to require many more conditions on the smoothness of the data of the problem.

In our work [3], we consider the following problem for a nonlinear equation (nonlinearity due to the function $g(t,x,y,u)$).

https://wordlyknowledge.uz/index.php/gpble
Qabul qilingan sana: 07.12.2025
Nashr etilgan sana: 30.01.2026

Let $Q_T=(0,T)\times G\times(0,\pi)$ and $D=G\times(0,\pi)$, where $G\subset\mathbb{R}^n$ is a bounded domain with a sufficiently smooth boundary. We consider the following inverse problem (note, $u(t,x,y)$ and $h(t,x)$ are unknowns):

$$\begin{cases} D_t^{\alpha}u-\Delta_x u-u_{yy}=g(t,x,y,u)+f(t,x,y)\cdot h(t,x), & (t,x,y)\in Q_T, \\ u(0,x,y)=\varphi(x,y), & (x,y)\in D, \\ u(t,x,y)=0, & x\in\partial G, t\in[0,T], y\in[0,\pi], \\ u(t,x,0)=u(t,x,\pi)=0, & t\in[0,T], x\in G, \end{cases}$$

with the overdetermination condition

$$\int_0^{\pi} u(t,x,y)\,\omega(y)\,dy=H(t,x), \quad t\in[0,T],\ x\in G.$$

Unlike previous works, this problem was investigated using the Galerkin method: For each $n\in\mathbb{N}$, we search for approximate solutions to the inverse problem:

$$u^n(t,x,y)=\sum_{j=1}^{n} c_j^n(t,x)\,\psi_j(y), \tag{19}$$

where the coefficients $c_1^n(t,x),\cdots,c_n^n(t,x)$ are unknown.

Let us formulate the conditions on the functions ω, f, g, and $H(t,x)$. Everywhere below, we will assume that

$$\omega\in \dot{W}_2^1(0,\pi)\cap W_2^2(0,\pi) \text{ and } \int_0^{\pi}\varphi(x,y)\,\omega(y)\,dy=H(0,x). \tag{20}$$

As for the function f, we will assume that it is continuous with respect to $(t,x,y)\in\overline{Q}_T$ and $(f(t,x,\cdot),\omega)\neq 0$ for all $(t,x)\in[0,T]\times\overline{G}$. In what follows, we will assume that the following estimate is fulfilled:

$$f_{\omega}(t,x,y):=\frac{f(t,x,y)}{(f(t,x,\cdot),\omega)}, \ \max_{\overline{Q}_T}|f_{\omega}(t,x,y)|\le f_M, \tag{21}$$

where f_M is some positive constant. The nonlinear function $g(t,x,y,u)$ must be continuous in arguments $(t,x,y)\in\overline{Q}_T$ (for any u) and satisfy the standard Lipschitz condition: for any u and v we assume that

https://wordlyknowledge.uz/index.php/gpble
Qabul qilingan sana: 07.12.2025
Nashr etilgan sana: 30.01.2026

$$|g(t,x,y,u)-g(t,x,y,v)|\le l_0|u-v|,\ (t,x,y)\in\overline{Q}_T, \tag{22}$$

where l_0 is some positive constant. As for the function $H(t,x)$, we will assume that $D_t^{\alpha}H(t,x),\ \Delta H(t,x)\in C([0,T]\times\overline{G})$.

Theorem 1.2 *Let us assume that all the conditions given above regarding the functions ω, f, g, and $H(t,x)$ are met. Then, the inverse problem has a unique weak solution. Moreover, the following estimates hold:*

$$\sup_{t\in[0,T]}\|u(t,\cdot,\cdot)\|^2_{L_2(D)}\le K_1, \tag{23}$$

$$\int_0^t\left(\|\nabla u(s,\cdot,\cdot)\|^2_{L_2(D)}+\|u_y(s,\cdot,\cdot)\|^2_{L_2(D)}\right)ds\le K_2, \tag{24}$$

$$\int_0^t\|D_s^{\alpha}u^n(s,\cdot,\cdot)\|^2_{L_2(D)}\,ds\le K_3, \tag{25}$$

$$\sup_{t\in[0,T]}||h(t,\cdot)||^2_{L_2(G)}\le K_4. \tag{26}$$

Here K_j, $j=1,2,3,4$, are some positive constants, depending on the data of the problem. It should be noted that, in the application of the Galerkin method to the inverse problem, unlike the previous case, no smallness conditions are required.

Conclusion

In this paper, an inverse coefficient identification problem for a diffusion-type equation with an integral overdetermination condition has been investigated. The considered problem arises naturally in models of anomalous diffusion and subdiffusion processes, where additional information in the form of integral constraints is available instead of classical boundary or initial data.

Under appropriate assumptions on the input data, the existence and uniqueness of a solution to the inverse problem have been established. The analysis is based on the reduction of the original inverse problem to an equivalent system involving a direct problem and an auxiliary equation for the unknown coefficient. By applying

https://wordlyknowledge.uz/index.php/gpble
Qabul qilingan sana: 07.12.2025
Nashr etilgan sana: 30.01.2026

functional analytic methods and a priori estimates, sufficient conditions guaranteeing the well-posedness of the problem are derived.

Moreover, stability estimates with respect to the given data have been obtained, demonstrating the continuous dependence of the recovered coefficient on the overdetermination condition. These results ensure the reliability of the proposed identification procedure and confirm that the inverse problem is not only solvable but also stable in an appropriate functional framework.

The theoretical findings presented in this work extend and generalize several known results on inverse problems for diffusion and subdiffusion equations. They can serve as a mathematical foundation for further studies, including numerical reconstruction algorithms and applications to real-world models in physics, engineering, and related applied sciences.

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