ON THE CONTINUITY OF OPERATORS IN BANACH SPACES

Kh.R. Umarov, S.I. Kuchkarova, M.S. Boykuzieva

Gulistan state university, Gulistan

e-mail: umarovhr@mail.ru

This paper presents problems illustrating the principle of proving the continuity of operators in Banach spaces.

Keywords: operator, Banach space, norm.

This paper is devoted to solving problems of operator continuity in Banach spaces, which are an important topic in functional analysis.

The article provides an example of one such problem.

Problem. Prove that the operator $A:C[-3;-1]\to C[-3;-1]$, $A(x)(t)=\sqrt[3]{x(t)}$ is continuous on its domain of definition.

Solution: Let's check whether this operator is linear, since if it is linear and bounded, its continuity can be immediately established. Let $x_1, x_2 \in C[-3;-1]$, then $A(x_1)(t)=\sqrt[3]{x_1(t)}$, $A(x_2)(t)=\sqrt[3]{x_2(t)}$. Therefore,

$$A(x_1)(t)+A(x_2)(t)=\sqrt[3]{x_1(t)}+\sqrt[3]{x_2(t)}, \text{ and } A(x_1+x_2)(t)=\sqrt[3]{x_1(t)+x_2(t)}.$$

Since

$$A(x_1)(t)+A(x_2)(t)\neq A(x_1+x_2)(t),$$

then the operator A is not linear, so we will prove its continuity according to Heine's definition.

Let a sequence $\{x_n(t)\}\subset C[-3;-1]$ be such that $x_n(t)\to x(t)\in C[-3,-1]$. Then the following estimate holds:

$$\|A(x_n)-A(x_0)\|_{C[-3;-1]}=\max_{-3\le t\le -1}\left|\sqrt[3]{x_n(t)}-\sqrt[3]{x_0(t)}\right|=$$

$$=\max_{-3\le t\le -1}\left|\frac{x_n(t)-x_0(t)}{\sqrt[3]{x_n^2(t)}+\sqrt[3]{x_n(t)x_0(t)}+\sqrt[3]{x_0^2(t)}}\right|=\max_{-3\le t\le -1}\frac{|x_n(t)-x_0(t)|}{\left|\sqrt[3]{x_n^2(t)}+\sqrt[3]{x_n(t)x_0(t)}+\sqrt[3]{x_0^2(t)}\right|}\le$$

$$\le\left(\max_{-3\le t\le -1}\frac{1}{\left|\sqrt[3]{x_n^2(t)}+\sqrt[3]{x_n(t)x_0(t)}+\sqrt[3]{x_0^2(t)}\right|}+1\right)\|x_n-x_0\|_{C[-3;-1]}\to 0 \text{ at } n\to\infty.$$

We obtain that the operator is continuous in the sense of Heine.

References

1. A.N.Kolmogorov, S.V.Fomin, Elements of the Theory of Functions and Functional Analysis; 7th ed. – Moscow: FIZMATLIT, 2006. – 572 pp.

2. L.A.Lyusternik, V.I.Sobolev, A Brief Course in Functional Analysis; – textbook, Moscow: Vyssh. shkola, 1982. – 271 pp.

3. V.I.Chilin, K.K.Muminov. Metric Spaces (textbook). Tashkent, 2010, 160 pp.