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EXIXTENCE CONDITIONS FOR EIGENVALUES OF THE LATTICE SCHRÖDINGER OPERATOR

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INTRODUCTION: In quantum modeling of interacting many-body systems for the manipulation of ultracold atoms and unique setting, coherent optical fields provide a strong tool because of their high-degree controllable parameters such as optical lattice geometry, dimension, particle mass, two-body potentials, temperature etc. (See [1-4]). The recent experimental and theoretical results show that integrating plasmonic systems with cold atoms, using optical potential fields formed from the near field scattering of light by an array of plasmonic nanoparticles, allows one to considerably increase the energy scales in the realization of Hubbard models and engineer effective long-range interaction in many body dynamics (See [5-7]). A several of numerical results for the bound state energies of one and two-particle systems in two adjacent 3D layers, connected through a window were presented in [8] and investigated the relation between the shape of a window and energy levels, as well as number of eigenfunction's nodal domains. In papers [9] and [10] were studied some spectral properties of the discrete Schrodinger operator with zero-range and short range attractive potentials, respectively.
In general, the Schrodinger operator $H(k), k \in \mathbb{T}^3$ associated to the lattice Hamiltonian H of two arbitrary particles with some dispersion relation and short range potential interaction acts in $L^2(\mathbb{T}^3)$ as [11]

$$H(k) = H_0(k) - V$$

where $H_0(k)$ is a multiplication operator by $\mathrm{E_k}(p) = \frac{1}{m_1}\varepsilon(p) + \frac{1}{m_2}\varepsilon(p-k)$ and V is integral operator with kernel $V(p,s) = V(p-s)$. In (See 8), several numerical results for the bound state energies of one and two-particle systems was presented in two adjacent 3D layers, connected through a window. The authors investigated the relation between the shape of a window and energy levels, as well as number of eigenfunction's nodal domains.

In [12], the existence conditions and positiveness of eigenvalues of the Schrodinger operator $H(k), k \in \mathbb{T}^d$ was studied with respect to the quasi-momentum k and the virtual level at the lower edge of essential spectrum.

The existence and absence of eigenvalues of the family $H(k)$ depending on the energy of interaction and quasi-momentum k were investigated in [13] and [14] for the cases

$$\varepsilon(p) = \sum_{i=1}^{3}(1-\cos 2p_i),\ V(p-s) = \sum_{\alpha=1}^{3}\mu_\alpha \cos(p_\alpha - s_\alpha)$$

and

$$\varepsilon(p) = \sum_{i=1}^{3}(1-\cos 2np_i),\ V(p-s) = \sum_{l=1}^{N}\sum_{i=1}^{3}\mu_{li}\cos l(p_i - s_i)$$

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respectively. The spectral properties of this operator $H(k)$ for the one dimensional case were studied in [15] and more general case in [16]. For general case $\varepsilon(p)$ satisfying some conditions and

$$V(p-s) = \mu_0 + \sum_{\alpha=1}^{3} \mu_\alpha \cos(p_\alpha - s_\alpha)$$

was investigated in [16].

2. STATEMENT OF THE MAIN RESULT

Let $\mathbb{T}^3$ be the three-dimensional torus and $L^2(\mathbb{T}^3)$ be a Gilbert space of square integrable functions on $\mathbb{T}^3$. Denote by $L^{2,o}(\mathbb{T}^3)$ the subspace of odd functions in $L^2(\mathbb{T}^3)$.

A Schrödinger operator $H_{\lambda\mu}$ acts in $L^{2,o}(\mathbb{T}^3)$, the image of the momentum in space is as follows:

$$H_{\lambda\mu} = H_0 - \mathrm{V}_{\lambda\mu},$$

where H_0 is the operator multiplication by the function $\varepsilon(q)$, i.e

$$(H_0 f)(q) = \varepsilon(q) f(q),\ f \in L^{2,o}(\mathbb{T}^3),$$

$\varepsilon(q) = \sum_{i=1}^{3}(1-\cos 2q_i)$, $V_{\lambda\mu}$ is an integral operator defined as

$$\left(V_{\lambda\mu} f\right)(q) = \lambda \sum_{i=1}^{3} \sin q_i \int_{\mathbb{T}^3} \sin t_i\, f(t)dt + \mu \sum_{i=1}^{3} \sin 2q_i \int_{\mathbb{T}^3} \sin 2t_i\, f(t)dt,$$

λ and μ are positive numbers representing an energy of the particles.

Lemma 2.1 The operator $H_{\lambda\mu}$ acts in $L_2^o(\mathbb{T}^3)$ and it is linear, bounded, self-adjoint operator.

Lemma 2.2 The operator $\mathrm{V}_{\lambda\mu}$ is non-negative, i.e. for arbitrary $f \in L^{2,o}(\mathbb{T}^3)$ $\left(\mathrm{V}_{\lambda\mu} f, f\right) \geq 0$.

Since $\varepsilon(q)$ is continuous function on $\mathbb{T}^3$, we have the following equations:

$$\varepsilon_{min} = \min_{q\in\mathbb{T}^3} \varepsilon(q) = 0,$$

$$\varepsilon_{max} = \max_{q\in\mathbb{T}^3} \varepsilon(q) = 6.$$

Note that by the Weyl theorem [2] on the essential spectrum the essential spectrum $\sigma_{ess}\left(h_{\lambda\mu}(k)\right)$ of the operator $h_{\lambda\mu}(k)$ coincides with the spectrum of the unperturbed operator $h_0(k)$. Therefore we can give the following lemma

Lemma 2.3 The following equalities

$$\sigma_{ess}\left(H_{\lambda\mu}\right) = \sigma_{ess}(H_0) = \sigma(H_0) = [0,6]$$

hold.

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Note that

$$\sigma_{ess}(H_{\lambda\mu}) = \sigma_{ess}(H_0) = \sigma(H_0) = [0,6]$$

$$\sigma(H_{\lambda\mu}) \cap (6, +\infty) = \emptyset.$$

Lemma 2.4 For any $z \in \mathbb{C} \setminus [\varepsilon_{min}, \varepsilon_{max}]$ the equalities

$$\int_{\mathbb{T}^3} \frac{\sin^2 t_1}{\varepsilon(t) - z} dt = \int_{\mathbb{T}^3} \frac{\sin^2 t_2}{\varepsilon(t) - z} dt = \int_{\mathbb{T}^3} \frac{\sin t_3}{\varepsilon(t) - z} dt,$$

$$\int_{\mathbb{T}^3} \frac{\sin^2 2t_1}{\varepsilon(t) - z} dt = \int_{\mathbb{T}^3} \frac{\sin^2 2t_2}{\varepsilon(t) - z} dt = \int_{\mathbb{T}^3} \frac{\sin 2t_3}{\varepsilon(t) - z} dt$$

hold.

Let $L^{2,o}(\mathbb{T}^3)$ be a space of odd function in $L^2(\mathbb{T}^3)$. We introduce the following subspaces

$$L^{2,eeo}(\mathbb{T}^3) = \{f \in L^{2,o}(\mathbb{T}^3): f(p_1, p_2, p_3) = f(-p_1, p_2, p_3) = f(p_1, -p_2, p_3) = -f(p_1, p_2, -p_3)\}$$

$$L^{2,eoe}(\mathbb{T}^3) = \{f \in L^{2,o}(\mathbb{T}^3): f(p_1, p_2, p_3) = f(-p_1, p_2, p_3) = -f(p_1, -p_2, p_3) = f(p_1, p_2, -p_3)\}$$

$$L^{2,oee}(\mathbb{T}^3) = \{f \in L^{2,o}(\mathbb{T}^3): f(p_1, p_2, p_3) = -f(-p_1, p_2, p_3) = f(p_1, -p_2, p_3) = f(p_1, p_2, -p_3)\}$$

$$L^{2,ooo}(\mathbb{T}^3) = \{f \in L^{2,o}(\mathbb{T}^3): f(p_1, p_2, p_3) = -f(-p_1, p_2, p_3) = -f(p_1, -p_2, p_3) = -f(p_1, p_2, -p_3)\}.$$

Lemma 2.5 The equality

$$L^{2,o}(\mathbb{T}^3) = L^{2,eeo}(\mathbb{T}^3) \oplus L^{2,eoe}(\mathbb{T}^3) \oplus L^{2,oee}(\mathbb{T}^3) \oplus L^{2,ooo}(\mathbb{T}^3)$$

holds.

We consider the integral operators $V^{\theta}_{\lambda\mu}, \theta \in \{eeo, eoe, oee, ooo\}$ acting in $L^2(\mathbb{T}^2)$ as

$$(V^{eeo}_{\lambda\mu} f)(p_1, p_2, p_3) = \lambda \sin p_3 \int_{\mathbb{T}^3} \sin t_3 f(t_1, t_2 t_3) dt + \mu \sin 2p_3 \int_{\mathbb{T}^3} \sin 2t_3 f(t_1, t_2 t_3) dt,$$

$$(V^{eoe}_{\lambda\mu} f)$$

$$(p_1, p_2, p_3) = \lambda \sin p_2 \int_{\mathbb{T}^3} \sin t_2 f(t_1, t_2 t_3) dt + \mu \sin 2p_2 \int_{\mathbb{T}^3} \sin 2t_2 f(t_1, t_2 t_3) dt,$$

$$(V^{oee}_{\lambda\mu} f)(p_1, p_2, p_3) = \lambda \sin p_1 \int_{\mathbb{T}^3} \sin t_1 f(t_1, t_2 t_3) dt + \mu \sin 2p_1 \int_{\mathbb{T}^3} \sin 2t_1 f(t_1, t_2 t_3) dt,$$

$$(V^{ooo}_{\lambda\mu} f)(p_1, p_2, p_3) = 0.$$

Note that

$$V^{\theta}_{\lambda\mu}: L^2(\mathbb{T}^2) \to L^{2,\theta}(\mathbb{T}^2), \quad \theta \in \{eeo, eoe, oee, ooo\}.$$

Let

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$H_{\lambda\mu}^{\theta} = H_0 + V_{\lambda\mu}^{\theta}, \ \theta \in \{eeo, eoe, oee, ooo\}$.

Lemma 2.6 The operator $H_{\lambda\mu}^{\theta}$, $\theta \in \{eeo, eoe, oee, ooo\}$ is invariant with respect to subspace $L^{2,\theta}(\mathbb{T}^3)$, $\theta \in \{eeo, eoe, oee, ooo\}$.

The Lemmas 2.5 and 2.6 gave the following lemma

Lemma 2.7 The equality

$$\sigma(H_{\lambda\mu}) = \sigma(H_{\lambda\mu}^{eeo}) \cup \sigma(H_{\lambda\mu}^{eoe}) \cup \sigma(H_{\lambda\mu}^{oee}) \cup \sigma(H_{\lambda\mu}^{ooo})$$

holds.

Where

$$\Delta^{\theta}(\lambda, \mu: z) = \left(1 - \lambda \int_{\mathbb{T}^3} \frac{\sin^2 t_1}{\varepsilon(t) - z} dt\right)^3 \left(1 - \mu \int_{\mathbb{T}^3} \frac{\sin^2 2t_1}{\varepsilon(t) - z} dt\right)^3.$$

Theorem 2.1 The number $z \in \mathbb{C} \setminus [\varepsilon_{min}, \varepsilon_{max}]$ is an eigenvalue of the operator $H_{\lambda\mu}^{\theta}$, $\theta \in \{eeo, eoe, oee, ooo\}$ if and only if

$$\Delta^{\theta}(\lambda, \mu: z) = 0$$

the fulfillment of equality is necessary and sufficient.

Lemma 2.8 $\Delta^{\theta}(\lambda, \mu: z)$ function $(-\infty, \varepsilon_{min})$ is monotonically decreasing in the interval and $\Delta^{\theta}(\lambda, \mu: z)$ if $z \to -\infty$.

We introduce the following numbers

$$\lambda_0^1 = \left(\int_{\mathbb{T}^3} \frac{\sin^2 t_1}{\varepsilon(t)} dt\right)^{-1} = \lambda_0^2 = \lambda_0^3,$$

$$\mu_0^1 = \left(\int_{\mathbb{T}^3} \frac{\sin^2 2t_1}{\varepsilon(t)} dt\right)^{-1} = \mu_0^2 = \mu_0^3,$$

$H_{\lambda\mu}$ spectrum of the operator (λ, μ) – to study depending on the parameter, we divide the (λ, μ) – plane into the following areas.

$$C_0^+ = \{(\lambda, \mu) \in \mathbb{R}^2: \lambda < \lambda_0^1\}$$

$$C_1^+ = \{(\lambda, \mu) \in \mathbb{R}^2: \lambda > \lambda_0^1\}$$

$$S_0^+ = \{(\lambda, \mu) \in \mathbb{R}^2: \mu < \mu_0^1\}$$

$$S_1^+ = \{(\lambda, \mu) \in \mathbb{R}^2: \mu > \mu_0^1\}$$

Theorem 2.2.

(i) Let $(\lambda, \mu) \in C_1^+ \cap S_1^+$. Then $H_{\lambda\mu}$ has two eigenvalues with multiplicity three lying on the left side of the essential spectrum of the operator $H_{\lambda\mu}$.

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(ii) Let $(\lambda,\mu) \in C_1^+ \cap S_1^+$. Then $H_{\lambda\mu}$ has one eigenvalue with multiplicity three lying on the left side of the essential spectrum of the operator $H_{\lambda\mu}$.

(iii) Let $(\lambda,\mu) \in C_1^+ \cap S_1^+$. Then $H_{\lambda\mu}$ has one eigenvalues with multiplicity three lying on the left side of the essential spectrum of the operator $H_{\lambda\mu}$.

(iv) Let $(\lambda,\mu) \in C_0^+ \cap S_0^+$ the operator $h_r^o(k)$ has no eigenvalue lying to the left.

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