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Pi strategy and its applications for the case where controls in differential games have integral constraints

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Abstract: This article considers the construction of a parallel pursuit strategy in simple differential games and its application to solve chase-escape problems. General information about simple differential games is presented throughout the article. It contains information about the basic concepts of the theory of differential games such as strategy, problems of chasing and escaping in integral, geometric, integral-geometric and other bounded differential games. In addition, in order to demonstrate the processes of solving the problem in classical games, the solution of a problem with an integral limit is shown in full.

Аннотация:В данной статье рассматривается построение стратегии параллельного преследования в простых дифференциальных играх и ее применение для решения задач погони-убегания.В статье представлены общие сведения о простых дифференциальных играх. Содержит информацию об основных понятиях теории дифференциальных игр, таких как стратегия, задачи преследования и побега в интегральных, геометрических, интегро-геометрических и других ограниченных дифференциальных играх. Кроме того, для демонстрации процессов решения задачи в классических играх решение задачи с интегральным пределом показано полностью.

Key words: Differential game, integral limit, chase problem, escape problem, catch time, escape time.

Ключевые слова: Дифференциальная игра, интегральный предел, задача погони, задача побега, время поимки, время побега.

1.Introduction.

Differential games are one of the important branches of mathematical management theory. The theory of differential games develops the ideas and methods of optimal control theory. The basic mathematical tool of the theory of differential games is the theory of ordinary differential equations. With the help of differential equations, a number of important laws of physics, mechanics, chemistry, biology, astronomy, economics, management theory are written and researched. The theory of differential games is a mathematical theory of conflicting situations in human activity and nature, explained by differential equations. In the theory of differential games, in addition to the research of conflicting situations, the issues of optimal management complicated by grid barriers are also considered.

Differential games are a complex mathematical concept due to the need to take into account factors such as dynamism, controllability, conflict, optimality, informativeness. Such factors require the development of special approaches to the theory of differential games. Such an approach depends primarily on solving chase-escape problems, which is an important branch of differential game theory.

2.Materials and methods.

In the scientific results of most chase-escape problems of the theory of differential games, the control functions are selected from the classes limited by the limit of assignment.The basic concepts of differential game theory depend on sets, the theory of mathematical analysis, the theory of differential equations, the theory of optimal control, and the basic concepts of the theory of linear algebra. In the theory of differential games, the concept of strategy occupies a central place. In the theory of differential games, a definition is given based on the nature of the problem.

3. Results and discussion.

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Chase problem for one integral bounded case

Let there be P chasing and E escaping objects in the space. Let their equations of motion be as follows:

$$P: \quad \dot{x} = u \qquad x(0) = x_0 \qquad (1)$$

$$E: \quad \dot{y} = v \qquad y(0) = y_0 \qquad (2)$$

U and V velocity parameters should be chosen as measurable functions in the Lebegue sense and satisfy the following constraints, respectively:

$$\int_0^t |u(s)|^2 ds \le \alpha t \text{ almost all } t \quad 0 \qquad (3)$$

$$\int_0^t |v(s)|^2 ds \le \beta t \text{ almost all } t \quad 0 \qquad (4)$$

(3) The meaning of bounding is called energy bounding and means that the speed of the chaser decreases with time. However, it can be proved that the chaser reaches an arbitrary point in space.

If the chaser is chosen to be an arbitrary function satisfying the constraint (3), using it

$$x(t) = x_0 + \int_0^t U(s)ds \qquad (5)$$

a trajectory is formed. From this

$$|x(t) - x_0| = \left| \int_0^t U(s)ds \right| \quad \int_0^t U(s)ds \quad \sqrt{\int_0^t |U(s)|^2 ds} \sqrt{\int_0^t ds} \quad \sqrt{t\rho_0}$$

$$|x(t) - x_0| \quad \sqrt{t\rho_0} = \mu(t) \quad x(t) \quad S_{\mu(t)}(x_0)$$

The chaser's goal is limited even when the fugitive acts voluntarily at time $t<\infty$

$$x(t) = y(t$$

achieve equality.

And the goal of the escaper is at an arbitrary time $t\ge 0$

$$x(t) \ne y(t)$$

achieve a relationship.

Let their controls have the following limitations:

$$\int_0^t |u(s)|^2 ds \le \alpha t \qquad (3)$$

$$\int_0^t |v(s)|^2 ds \le \beta t \qquad (4)$$

Definition-1: If constraints (3) and (4) are given in the problem (1)-(2), then such a problem is called a differential game problem with an integral constraint.

The physical meaning of the conditions (3) and (4) means the change of some energy. The more energy is spent, the faster the process can be completed. If the control functions are small, the process can last longer.

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Explanation: The physical meaning of the problem is that in the case of integral limitation, the chaser does not run out of energy until he catches the fugitive, that is, it ends when he catches him.

$u-$ velocity vector, $P-$ control parameter and $u(\cdot)\colon [0;\infty) \to R^n$ is a measurable function in the Lebesgue sense reflecting .

(3) we denote the class of u-measurable functions satisfying the boundedness by U_{G_r}

$v(\cdot)\colon [0;\infty) \to R^n$is a measurable function in the Lebesgue sense reflecting .

$v(\cdot)$-velocity vector, E-control parameter of the object.

(4) we denote the class of u measurable functions satisfying the boundedness by V_{G_r}.

Definition-2: $u(\cdot)\epsilon U_{G_r}$and $v(\cdot)\epsilon V_{G_r}$ it is called the control of chasing and evading objects according to the measurable functions.

Definition-3:$\big(x_0; u(\cdot)\big), u(\cdot)\epsilon U_{G_r}$ solution of equation (1) for the pair

$$x(t) = x_0 + \int_0^t u_I(s)ds\, t \geq 0$$

is called the chaser's trajectory.

Definition-4:$\big(y_0; v(\cdot)\big), v(\cdot)\epsilon V_{G_r}$ solution of equation (2) for the pair

$$y(t) = y_0 + \int_0^t v_I(s)ds\, t \geq 0$$

is called the runner's trajectory.

First, we see the problem of catching, that is, even when the escaper chooses an arbitrary function satisfying the constraint (4), for the pursuer, it is necessary to chase with a strategy $\exists$ at some time t^*<∞ $x(t^*) = y(t^*)$ equals.

Construction of the strategy.

We assume that at some time t≥0 the objects are in the position x(t) and y(t), respectively. Let their remaining resources at this time be $\rho(t)$ and $\sigma(t)$.The following formulas for resources and cases are appropriate here:

$$x(t) = x_0 + \int_0^t u_I(v(s))ds\ \rho(t) = \alpha t - \int_0^t |u(s)|^2 ds$$

$$y(t) = y_0 + \int_0^t v(s)ds \qquad \sigma(t) = \beta t - \int_0^t |v(s)|^2 ds$$

From position X, the pursuer should walk in such a direction that if the objects meet at some point ω, the remaining resources of the pursuer and the remaining resources of the escaper are equal at this point.

$$\alpha T - |u|^2 T = \beta T - |v|^2 T$$

The meaning of this equality is that resources must remain the same when objects go to point ω.

$$z_0 = x_0 - y_0$$

$$uT + z_0 = vT$$

$$u = v - \frac{z_0}{T} = v - \frac{z_0}{|z_0|} \times \frac{|z_0|}{T} = v - \xi_0 \lambda$$

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$$\begin{cases} u(v) = v - \lambda\xi_0 \\ |u|^2 = \alpha - \beta + |v|^2 \end{cases}$$

$$|u|^2 = |v|^2 - 2\langle v;\xi_0\rangle\lambda + \lambda^2$$

$$|v|^2 - 2\langle v;\xi_0\rangle\lambda + \lambda^2 = \alpha - \beta + |v|^2$$

$$\lambda^2 - 2\langle v;\xi_0\rangle\lambda - \delta = 0$$

$$\lambda_{1,2} = \frac{2\langle v;\xi_0\rangle \pm \sqrt{4\langle v;\xi_0\rangle^2 + 4\delta}}{2} = \langle v;\xi_0\rangle \pm \sqrt{\langle v;\xi_0\rangle^2 + \delta}$$

we get λ_{max}:

$$\lambda_{max} = \langle v;\xi_0\rangle + \sqrt{\langle v;\xi_0\rangle^2 + \delta}$$

$u(v) = v - \lambda_I(v)\xi_0$ (3.2.5) $\lambda_I(v) = \max\{\langle v;\xi_0\rangle + \sqrt{\langle v;\xi_0\rangle^2 + \delta}\}$

Definition-5: (5) strategy is called π-strategy with integral limitation.

Definition-6: Let it be $\alpha > \beta$. The strategy of the chaser in the differential game (1)-(4) is expressed in the following equation:

$u(v) = v - \lambda\xi_0$ (5)

Here - $\xi_0 = \frac{z_0}{|z_0|}$, $\lambda(v) = \langle v;\xi_0\rangle + \sqrt{\langle v;\xi_0\rangle^2 + \delta}$

$$\delta = \alpha - \beta$$

Theorem1: The following is for the chaser in all $\alpha > \beta$

$$|u|^2 = |v|^2 + \delta$$

the condition is met.

Proof: $u(v) = v - \lambda\xi_0$

$$|u|^2 = |v|^2 - 2\langle v;\xi_0\rangle\lambda + \lambda^2 = |u|^2 = |v|^2 + \lambda(\lambda - 2\langle v;\xi_0\rangle)$$
$$= |v|^2 + \left(\langle v;\xi_0\rangle + \sqrt{\langle v;\xi_0\rangle^2 + \delta}\right)\left(\langle v;\xi_0\rangle + \sqrt{\langle v;\xi_0\rangle^2 + \delta} - 2\langle v;\xi_0\rangle\right)$$
$$= |v|^2 + \left(\langle v;\xi_0\rangle + \sqrt{\langle v;\xi_0\rangle^2 + \delta}\right)\left(\sqrt{\langle v;\xi_0\rangle^2 + \delta} - \langle v;\xi_0\rangle\right)$$
$$= |v|^2 + \langle v;\xi_0\rangle^2 + \delta - \langle v;\xi_0\rangle^2 = |v|^2 + \delta$$

The theorem is proved.

A solution to the problem of retention.

Theorem-2: Suppose $\alpha > \beta$. Then in the differential game (1)-(4), strategy (5) is said to be successful for the chaser, or the chaser ends the game using strategy (5).

Time until the end of the game $[0; t_I]$

Here - $t_I = \frac{|z_0|}{\sqrt{\alpha} - \sqrt{\beta}}$

Proof: Let the escaper choose an arbitrary strategy. Let the pursuer use strategy (5). Then, using equation (1)-(2), we can write the following equation:

$$x(t) = x_0 + \int_0^t u_I(v(s))ds$$

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$$y(t) = y_0 + \int_0^t v(s)ds$$

$$z(t) = z_0 + \int_0^t [u_1(v(s)) - v(s)]ds = z_0 - \int_0^t \lambda(v(s))\xi_0 ds = z_0$$
$$- \frac{z_0}{|z_0|}\int_0^t \lambda(v(s))ds = z_0\left(1 - \frac{1}{|z_0|}\int_0^t \lambda(v(s))ds\right) = \Lambda(t, v(\cdot))$$

$$\Lambda(t, v(\cdot)) = 1 - \frac{1}{|z_0|}\int_0^t \lambda(v(s))ds = 1 - \frac{1}{|z_0|}\int_0^t (\langle v; \xi_0\rangle + \sqrt{\langle v; \xi_0\rangle^2 + \delta})ds$$

$\langle v; \xi_0\rangle + \sqrt{\langle v; \xi_0\rangle^2 + \delta}$ - we check the function for monotonicity
:

$$f(x) = x + \sqrt{x^2 + \delta}$$

$$f'(x) = 1 + \frac{x}{\sqrt{x^2 + \delta}}$$

This function is monotonically increasing. Let's put the smallest value of this function:

$$\Lambda(t, v(\cdot)) \le 1 - \frac{1}{|z_0|}\int_0^t (-|v(s)| + \sqrt{|v(s)|^2 + \delta})ds$$

$$f(v(s)) = -|v(s)| + \sqrt{|v(s)|^2 + \delta}$$

$$f'(v(s)) = -1 + \frac{|v(s)|}{\sqrt{|v(s)|^2 + \delta}}$$

$$f''(v(s)) = \frac{\sqrt{|v(s)|^2 + \delta} - v(s)\frac{v(s)}{\sqrt{|v(s)|^2 + \delta}}}{|v(s)|^2 + \delta} = \frac{|v(s)|^2 + \delta - |v(s)|^2}{(|v(s)|^2 + \delta)^{3/2}} = \frac{\delta}{(|v(s)|^2 + \delta)^{3/2}} > 0$$

$f''(x) > 0$ if the condition is met, we can apply the following Jensen's inequality:
Jensen's inequality:

$$\int_0^t f(v(s))ds \ge t \cdot f\left(\frac{1}{t}\int_0^t f(v(s))ds\right)$$

$$\Lambda(t, v(\cdot)) = 1 - \frac{1}{|z_0|}\int_0^t [-|v(s)| + \sqrt{|v(s)|^2 + \delta}]ds \le 1 - \frac{t}{|z_0|}[-\frac{1}{t}\int_0^t |v(s)|ds +$$
$$\sqrt{(\frac{1}{t}\int_0^t |v(s)|ds)^2 + \delta}]$$

we apply the Cauchy-Buniakovsky inequality to $\int_0^t |v(s)|ds$

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$$\int_0^t f(s)\cdot g(s)ds \le \sqrt{\int_0^t f^2(s)ds)}\cdot\sqrt{\int_0^t g^2(s)ds)}$$

$$\int_0^t |v(s)|ds \le \sqrt{t}\cdot\sqrt{\int_0^t |v(s)|^2 ds} \le \sqrt{t}\cdot\sqrt{\beta t} = \sqrt{\beta}t$$

$$\Lambda\big(t,v(\cdot)\big) = 1-\frac{1}{|z_0|}\int_0^t [-|v(s)|+\sqrt{|v(s)|^2+\delta}]ds \le 1-\frac{t}{|z_0|}[-\frac{1}{t}\int_0^t |v(s)|ds + \sqrt{(\frac{1}{t}\int_0^t |v(s)|ds)^2+\delta}\ ] \le 1-\frac{t}{|z_0|}[-\frac{1}{t}\sqrt{\beta}t+\sqrt{\Big(\frac{1}{t}\sqrt{\beta}t\Big)^2+\delta} = 1-\frac{t}{|z_0|}[-\sqrt{\beta}+\sqrt{\beta+\alpha-\beta}] = 1-\frac{t}{|z_0|}[\sqrt{\alpha}-\sqrt{\beta}] = 0$$

It follows that the holding time is equal to

$t_I = \frac{|z_0|}{\sqrt{\alpha}-\sqrt{\beta}}$.

The theorem is proven.

We check the validity:

$$|u|^2 = |v|^2+\delta$$

$$\int_0^{t_I} |u(s)|^2 ds = \int_0^{t_I} |v(s)|^2 ds + \delta \le \beta t + \alpha t_I - \beta t = \alpha t_I$$

$\int_0^t |u(s)|^2 ds \le \alpha\, t_I$

The problem of detention has been fully resolved.

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